GROUP THEORY 2025 - 26, EXERCISE SHEET 1

Exercise 1. *The Permutation Group S_n*

(1) Recall from *structures fondamentales* that every element of S_n can be represented as a product of disjoint cycles. We start off with a quick warm-up of the cycle notation.

   (a) Write the following element of the group S_5 in cycle notation :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

   (b) Compute the product $(1\,3\,4\,5)\,(2\,6) \cdot (1\,6\,5\,2)\,(3\,4)$ in S_6.

   (c) Compute the inverse of the permutation $(1\,3\,4\,5)\,(2\,6)$ in the group S_6.

(2) Let p be a prime number and let $a \in S_n$ such that $a^p = 1$, then show that a can be written as a product of disjoint p - cycles (that is cycles of length p).

(3) Recall that a transposition in S_n is a permutation that can be written as $(i\,j)$ with $1 \leq i < j \leq n$.

   (a) Show that every permutation in S_n can be written as a product of transpositions.

   (b) Let τ_i and π_i be some transpositions in S_n such that

$$\tau_1 \cdot \tau_2 \cdot \ldots \cdot \tau_k = \pi_1 \cdot \pi_2 \cdot \ldots \cdot \pi_l.$$

   Show that l and k have the same parity, that is $l - k = 0 \bmod 2$.

   (c) Permutations that can be written as a product of an odd (respectively even) number of transpositions are called odd (respectively even) permutations. Show that an m - cycle in S_n is an odd permutation if and only if m is an even integer.

Exercise 2. *The Dihedral Group D_{2n}*

Recall the definition of the dihedral group.

(1) Show that the set D_{2n} endowed with the function composition law forms a group. Is it a subgroup of S_n ?

(2) Show that there exists an isomorphism $D_6 \cong S_3$ and deduce that D_6 is not abelian;

(3) Find two elements $r, s \in D_6$ such that every element $x \in D_6$ can be written as a composition of r and s.

(4) Generalise the above exercise and find two elements $r, s \in D_{2n}$ such that every element $x \in D_{2n}$ can be written as a composition of r and s. This means that D_{2n} is generated by two elements. Later on in the course we will see that D_{2n} has a *presentation* with generators r and s.

(5) Show that D_{2n} is not abelian for all $n \geq 3$.

Exercise 3. *Some more examples of groups.*

(1) Show that the set $G := \{1, -1, i, -i\}$ endowed with complex multiplication forms a group isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
(2) Let K be the set of all complex numbers z such that $|\, z \,| = 1$. Verify that K is a group under complex multiplication.
(3) Let H be the set of all complex n^{th} roots of unity, that is all solutions to the equation $X^n - 1 = 0$ with $X \in \mathbb{C}$. Prove that G is a group under complex multiplication which is isomorphic to the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Exercise 4. *Group Actions*

Let $\cdot : G \times X \to X$ be an action of a group G on a set X and let $x \in X$ be an element. We define the stabiliser of x (with respect to the action $\cdot$) as follows:

$$\mathrm{Stab}_G(x) := \{g \in G |\ g \cdot x = x\} \subseteq G.$$

(1) Show that $\mathrm{Stab}_G(x)$ is a subgroup of G.

For the following actions, determine the stabiliser groups of the indicated $x \in X$:

(2) Let $G = S_n$ acting on $X = \{1, \ldots, n\}$ with the permutation action. Solve for all $x \in X$.
(3) For a field K, let $G = GL_n(K)$ the group of invertible $n \times n$ matrices endowed with matrix multiplication, and let $X := M_{n\times n}(K)$ be the set of $n \times n$ matrices. Consider the multiplication action $A \cdot B = AB$ for $A \in G$ and $B \in X$. Solve for $x = E_{i,j}$ and all $x \in GL_n(K)$. Recall that for $i, j \in \{1, ..., n\}$, the matrix $E_{i,j} \in M_{n\times n}(K)$ has 1 as the $(i, j)^{th}$ and 0 for all other entries.
(4) Again, let $G = GL_n(K)$ be the group of invertible $n \times n$ matrices over the field K and let $X := M_{n\times n}(K)$ be the set of $n \times n$ matrices. Consider the conjugation action $A \cdot B := ABA^{-1}$. Solve first for x a scalar matrix, and then for x a diagonal matrix with *distinct* entries.
(5) Given a group G, the left multiplication action $G \times G \to G$ is defined by $g \cdot g' = gg'$; and the conjugation action is defined as $g \bullet g' = gg'g^{-1}$. For $G = S_n$ and the aforementioned actions, compute the stabiliser groups for any transposition $(ij) \in S_n$ and for any 3-cycle $(ijk) \in S_n$. Also compute the stabilisers for the above mentioned actions when $G = \mathbb{Z}/n\mathbb{Z}$ for all $x \in \mathbb{Z}/n\mathbb{Z}$.
(6) Let $G = C_2 = \{1, -1\}$ a multiplicative cyclic group of order 2 and $X = \mathbb{R}$ with the action given by multiplication. Solve for all $x \in X$.

Exercise 5. *Some Geometric Actions*

(1) Let G be the group of rotations of a cube, show that $G \cong S_4$ by considering the natural action of G on the 4 long diagonals of the cube.
(2) Recall that the group D_{2n} has a defining action on the set of vertices of a regular n - gon. The aim of this exercise is to show that D_{12} can be realised as a subgroup of S_5 by considering an appropriate geometric action:
Let $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ denote the set of vertices of a regular hexagon in cyclic order.

By T_1 and T_2 we denote the triangles with vertex sets $\{a_1, a_3, a_5\}$ and $\{a_2, a_4, a_6\}$ respectively. Also by D_1, D_2, D_3 we denote the long diagonals of the hexagon, that is the line segments $\{a_1, a_4\}$, $\{a_2, a_5\}$, $\{a_3, a_6\}$ respectively. We define

$$S := \{T_1, T_2, D_1, D_2, D_3\}.$$

Observe that the natural action of D_{12} on a regular hexagon defines an action on S and show that the corresponding group homomorphism

$$D_{12} \to S_5$$

is injective.